

Probability of color singlet chain states in e^+e^- annihilation

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Abstract

We use the method of the color effective Hamiltonian to study the structure of color singlet chain states in $N_c = 3$ and in the large N_c limit. In order to obtain their total fraction when N_c is finite, we illustrate how to orthogonalize these non-orthogonal states. We give numerical results for the fraction of orthogonalized states in $e^+e^- \rightarrow q\bar{q}gg$. With the help of a diagram technique, we derive their fraction up to $O(1/N_c^2)$ for the general multigluon process. For large N_c the singlet chain states correspond to well-defined color topologies. Therefore we may expect that the fraction of non-color-singlet-chain states is an estimate of the fraction of events where color reconnection is possible. In the case of soft gluon bremsstrahlung, we give an explicit form for the color effective Hamiltonian which leads to the dipole cascade formulation for parton showering in leading order in N_c . The next-to-leading order corrections are also given for $e^+e^- \rightarrow q\bar{q}g_1g_2$ and $e^+e^- \rightarrow q\bar{q}g_1g_2g_3$.

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I. INTRODUCTION

Hadronic processes in various high energy collisions are generally described in terms of two distinct phases: the perturbative phase and the non-perturbative hadronization one. The perturbative phase is well described by perturbative QCD (PQCD) while the hadronization phase cannot be described from first principle and can only be described by phenomenological models, e.g. the Lund string model [1,2] or the cluster model [3]. These

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two phases are usually assumed to be well separated from each other. It is believed that the cross section for the hadronic process is fully determined by the perturbative phase, while in the hadronization phase a definite hadron state is chosen with total probability 1. In both phases, however, the large N_c (the number of color) approximation is implied, which reduces the possible interference effects. A color charge of one parton is specifically connected to its anti-color in an accompanying parton, and with infinitely many colors the probability that two (or more) partons have the same color is zero [4]. So here enters the phenomenological color flow method (CFM) commonly used in the Lund model and the cluster model which implies assigning the color connection for the final parton system [2]. In these models, for every $e^+e^- \rightarrow q\bar{q} + ng$ event, a neutral color flow is definitely determined and begins at the quark, connects each gluon one by one in a certain order, and ends at the antiquark. Each flow piece spanned between two partons is color-neutral and its hadronization is treated in a way similar to an independent $q\bar{q}$ singlet system. The present hadronization models work well, which shows that the CFM or the large N_c limit reflects some feature of the real world.

Recently we proposed a strict PQCD method to study the color structure of a multiparton system. This method is called the method of color effective Hamiltonian which is constructed from the PQCD invariant amplitude. In this method a gluon is treated as an exact color octet, not a bi-color or nonet. The goal of this paper is to study the structure of the color singlet chain state (SCS or chain state) using this method for $N_c = 3$ and in the large N_c limit.

The paper is organized as follows. In section II, we outline the effective Hamiltonian method. In section III, we define SCS, and introduce a diagram method which is very efficient in calculating the color inner product of any two color states. In section IV, we calculate the fraction of SCS for $N_c = 3$. Since these states are not orthogonal to each other when N_c is finite, we must find orthogonalized states to obtain a correct result. We introduce recipes for orthogonalization and use the orthogonalized states to calculate their fraction. As an example, we give the numerical result for the fraction of orthogonalized SCS in the process of e^+e^- annihilation into quark, antiquark and two gluons. Section V presents properties of SCS in the large N_c limit. With the help of the diagram technique, we give their fraction up to $O(1/N_c^2)$ for general multigluon process. In section VI, we give an explicit form for the momentum function D in H_c for the case of soft gluon bremsstrahlung. We show in the large N_c limit the consistency of H_c with the dipole cascade formulation for parton showering. We also discuss the next-to-leading order corrections for $e^+e^- \rightarrow q\bar{q}g_1g_2$ and $e^+e^- \rightarrow q\bar{q}g_1g_2g_3$.

II. COLOR EFFECTIVE HAMILTONIAN

The QCD Lagrangian describes the $SU(3)_c$ gauge interaction of gluon fields A_μ^α ($\alpha = 1, \dots, 8$) and quark fields q with three colors (R, Y, B). By redefining Gell-Mann Matrices and gluon fields, we obtain eight new fields ($X_\mu^{12}, X_\mu^{21}, X_\mu^{23}, X_\mu^{32}, X_\mu^{31}, X_\mu^{13}, X_\mu^3, X_\mu^8$) which couple to color combinations ($Y\bar{R}, R\bar{Y}, B\bar{Y}, Y\bar{B}, R\bar{B}, B\bar{R}, R\bar{R}/Y\bar{Y}, R\bar{R}/Y\bar{Y}/B\bar{B}$) respectively. Hence we write the QCD Lagrangian in a form with the quark-gluon interaction term showing clear color significance. This provides a strict formulation for calculating the fraction of color singlets for a multiparton system at the tree level from PQCD [5,6]. For the process $e^+e^- \rightarrow q\bar{q} + ng$, the essential part of the formulation is to exploit the color effective

Hamiltonian H_c to compute the amplitude $\langle f | H_c | 0 \rangle$ for a certain color state $|f\rangle$. The color effective Hamiltonian H_c is found from the invariant amplitude $M_{ab}^{\alpha_1 \alpha_2 \dots \alpha_n}$ to be:

$$\begin{aligned} H_c &= \sum_P (T^{\alpha_{P(1)}} T^{\alpha_{P(2)}} \dots T^{\alpha_{P(n)}})_{ab} D^P \Psi_a^\dagger \Psi^{b\dagger} A_1^{\alpha_1\dagger} A_2^{\alpha_2\dagger} \dots A_n^{\alpha_n\dagger} \\ &= \sum_P (1/\sqrt{2})^n \text{Tr}(Q^\dagger G_{P(1)}^\dagger G_{P(2)}^\dagger \dots G_{P(n)}^\dagger) D^P \end{aligned} \quad (1)$$

where Ψ_a^\dagger and $\Psi^{b\dagger}$ are color operators for the quark and antiquark; $(Q^\dagger)_a^b = \Psi^{b\dagger} \Psi_a^\dagger$ is the nonet tensor operator; G_u^\dagger is the gluon's color octet operator; D^P is a function of parton momenta and its dependence on the order of partons is marked by P which denotes a permutation of parton labels $(1, 2, \dots, n)$. The color effective Hamiltonian is another expression for the S matrix, and is therefore not Hermitian. One can verify its validity by using H_c to calculate the matrix element for the process $e^+ e^- \rightarrow q\bar{q} + ng$, which returns the original $|M_{ab}^{\alpha_1 \alpha_2 \dots \alpha_n}|^2$. In order to make H_c useful, we can define color states for the parton system $q\bar{q} + ng$ independently. A color state can be defined in the color space where each gluon subspace is either $\underline{8}$ or a little larger $\underline{3} \otimes \underline{3}^*$. The effects of the unphysical "singlet-gluon" state, brought by enlarging the color space $\underline{8}$ to $\underline{3} \otimes \underline{3}^* (= \underline{1} \oplus \underline{8})$, can be eliminated by calculating the projection of the state on H_c . There are many ways of reducing the color space. Corresponding to each reduction recipe there is one set of orthogonal singlets spaces whose bases make up a complete and orthogonal set of color singlets. H_c has the property from unitarity, that the sum of the cross sections over all color singlets in a complete and orthogonal singlet set for the system $q\bar{q} + ng$, is equal to the total cross section of $e^+ e^- \rightarrow q\bar{q} + ng$ at the tree level.

III. SINGLET CHAIN STATES IN $Q\bar{Q} + NG$ SYSTEM

A chain state for a $q\bar{q} + ng$ system is made up of a chain of $(n+1)$ pieces. Each piece is a singlet formed by the color charge of one parton and its anti-charge of the other one, and its color structure is the same as a $q\bar{q}$ system. A neutral color flow is also composed of pieces, each of which is formed by the color charge of one parton and its anti-charge of the other one, but it is a neutral color state not necessarily a color singlet. A singlet chain state can be decomposed into neutral color flow states. There are $n!$ chain states which connect q via n gluons in a specified order to \bar{q} , where the order of gluons is denoted by a permutation of $(1, 2, \dots, n)$: $P(1), P(2), \dots, P(n)$. These states can be written as

$$(|f_i\rangle, i = 1, 2, \dots, n!) \equiv \{N_c^{-(n+1)/2} | 1_{qP(1)} 1_{P(1)P(2)} 1_{P(2)P(3)} \dots 1_{P(n)\bar{q}} \rangle\} \quad (2)$$

where $N_c^{-(n+1)/2}$ is a normalization factor. Any two different chain states, $|f_i\rangle$ and $|f_j\rangle$, are not orthogonal to each other, i.e. $\langle f_i | f_j \rangle \neq 0$. However, we can prove that any two different chain states are approximately orthogonal to each other to order $1/N_c^2$. As an example, we look at the inner product of two states as follows:

$$\begin{aligned} |f_1\rangle &= N_c^{-(n+1)/2} | 1_{q1} 1_{12} 1_{23} 1_{34} 1_{45} \dots 1_{i,i+1} \dots 1_{n\bar{q}} \rangle \\ |f_2\rangle &= N_c^{-(n+1)/2} | 1_{q1} 1_{13} 1_{32} 1_{24} 1_{45} \dots 1_{i,i+1} \dots 1_{n\bar{q}} \rangle \\ C_{12} &\equiv \langle f_1 | f_2 \rangle \\ &= N_c^{-(n+1)} \langle 1_{q1} 1_{12} 1_{23} 1_{34} 1_{45} \dots 1_{i,i+1} \dots 1_{n\bar{q}} | 1_{q1} 1_{13} 1_{32} 1_{24} 1_{45} \dots 1_{i,i+1} \dots 1_{n\bar{q}} \rangle \end{aligned}$$

The gluon order in $|f_1\rangle$ and $|f_2\rangle$ is $(12345 \cdots ii+1 \cdots n)$ and $(13245 \cdots ii+1 \cdots n)$ respectively. The only difference is that the position of g_2 and g_3 is interchanged in two states. We write the explicit form of $|f_1\rangle$ and $|f_2\rangle$ as:

$$\begin{aligned} |f_1\rangle &= N_c^{-(n+1)/2} \delta(a_0, b_1) \delta(a_1, b_2) \delta(a_2, b_3) \delta(a_3, b_4) \delta(a_4, b_5) \cdots \delta(a_i, b_{i+1}) \cdots \delta(a_n, b_0) \\ &\quad \left| \Psi_{a_0} \Psi^{b_1} \Psi_{a_1} \Psi^{b_2} \Psi_{a_2} \Psi^{b_3} \Psi_{a_3} \Psi^{b_4} \Psi_{a_4} \Psi^{b_5} \cdots \Psi_{a_i} \Psi^{b_{i+1}} \cdots \Psi_{a_n} \Psi^{b_0} \right\rangle \\ |f_2\rangle &= N_c^{-(n+1)/2} \delta(a'_0, b'_1) \delta(a'_1, b'_3) \delta(a'_3, b'_2) \delta(a'_2, b'_4) \delta(a'_4, b'_5) \cdots \delta(a'_i, b'_{i+1}) \cdots \delta(a'_n, b'_0) \\ &\quad \left| \Psi_{a'_0} \Psi^{b'_1} \Psi_{a'_1} \Psi^{b'_3} \Psi_{a'_3} \Psi^{b'_2} \Psi_{a'_2} \Psi^{b'_4} \Psi_{a'_4} \Psi^{b'_5} \cdots \Psi_{a'_i} \Psi^{b'_{i+1}} \cdots \Psi_{a'_n} \Psi^{b'_0} \right\rangle \end{aligned}$$

where a, b, a', b' are the color and anticolor indices, and $0, 1, 2, 3, \dots, n$ denote $q(\bar{q})$, g_1 , g_2 , g_3, \dots, g_n . Thus $\langle f_1 | f_2 \rangle$ is:

$$\begin{aligned} \langle f_1 | f_2 \rangle &= N_c^{-(n+1)} [\delta(a_0, b_1) \delta(a_1, b_2) \delta(a_2, b_3) \delta(a_3, b_4) \delta(a_4, b_5) \cdots \delta(a_i, b_{i+1}) \cdots \delta(a_n, b_0)] \\ &\quad \cdot [\delta(a'_0, b'_1) \delta(a'_1, b'_3) \delta(a'_3, b'_2) \delta(a'_2, b'_4) \delta(a'_4, b'_5) \cdots \delta(a'_i, b'_{i+1}) \cdots \delta(a'_n, b'_0)] \\ &\quad \cdot \prod_{u=0,1,\dots,n} \delta(a_u, a'_u) \delta(b_u, b'_u) \end{aligned} \quad (3)$$

When $\prod_{u=0,1,\dots,n} \delta(a_u, a'_u) \delta(b_u, b'_u)$ is contracted with the content in the second square bracket, the above equation becomes

$$\begin{aligned} \langle f_1 | f_2 \rangle &= N_c^{-(n+1)} [\delta(a_0, b_1) \delta(a_1, b_2) \delta(a_2, b_3) \delta(a_3, b_4) \delta(a_4, b_5) \cdots \delta(a_i, b_{i+1}) \cdots \delta(a_n, b_0)] \\ &\quad \cdot [\delta(a_0, b_1) \delta(a_1, b_3) \delta(a_3, b_2) \delta(a_2, b_4) \delta(a_4, b_5) \cdots \delta(a_i, b_{i+1}) \cdots \delta(a_n, b_0)] \\ &= N_c^{-(n+1)} \cdot N_c \cdot N_c^{n-2} = 1/N_c^2 \end{aligned} \quad (4)$$

We can use diagrams to visualize and simplify our calculation of inner products. Let us write the color and anticolor indices into two rows where the numbers in the first one are anticolor indices and those in the second are color ones. We draw a line between the number v in the first row and the number u in the second row if there is a $\delta(a_u, b_v)$ in $\langle f_i | f_j \rangle$. As a rule, we have $\langle f_i | f_j \rangle = N_c^{l-n-1}$ where l is the number of closed paths. We can verify eq.(4) by drawing the corresponding diagram. As an example of (4), we show the $l = 2$ case in Fig.1.

Generally, we can carry out the inner product of any two states by a diagram:

$$\begin{aligned} \langle f_i | f_j \rangle &= N_c^{-(n+1)} [\delta(a_0, b_{P\{1\}}) \delta(a_{P(1)}, b_{P(2)}) \delta(a_{P(2)}, b_{P(3)}) \cdots \delta(a_{P(i)}, b_{P(i+1)}) \cdots \delta(a_{P(n)}, b_0)] \\ &\quad \cdot [\delta(a_0, b_{P'\{1\}}) \delta(a_{P'(1)}, b_{P'(2)}) \delta(a_{P'(2)}, b_{P'(3)}) \cdots \delta(a_{P'(i)}, b_{P'(i+1)}) \cdots \delta(a_{P'(n)}, b_0)] \\ &= N_c^{-(n+1)} \cdot N_c^{n_1} \cdot N_c^{n_2} = N_c^{n_1+n_2-(n+1)} \end{aligned} \quad (5)$$

If there is a factor $\delta(a_{P(i)}, b_{P(i+1)}) \delta(a_{P'(j)}, b_{P'(j+1)})$ where $P(i) = P'(j)$ and $P(i+1) = P'(j+1)$, we get one factor N_c . n_2 denotes the number of such identical δ -symbols in two square brackets of (5). If we exclude the identical δ -symbols in the two brackets, the rest contains only different δ s. We denote the number of such δ s in one bracket as n_d . It can be demonstrated from the diagram that when n_d is odd, these different δ s form one loop, and when n_d is even they form two loops. n_1 denotes the number of loops formed by those different δ s and is $2/1$ for even/odd n_d . The maximum of $n_1 + n_2 - (n+1)$ is -2 , which occurs when n_2 reaches its maximum value $n-2$. In this case the only difference between the two states is that only two gluons have their positions interchanged, just what we see in (3) and (4). Hence we see that any two different chain states are approximately orthogonal to each other to the order of $1/N_c^2$.

IV. ORTHOGONALIZATION AND FRACTION OF SINGLET CHAIN STATES AT FINITE N_C

For finite N_c , as in Nature, the chain states in (2) are not orthogonal to each other. We cannot directly take the sum of each $|\langle f_i | H_c | 0 \rangle|^2$ to give the total fraction for chain states, because it would make the contribution from the overlapped part of any two different states counted multiply. In this section we find a set of orthogonal states based on $\{|f_i\rangle\}$.

For the process $e^+e^- \rightarrow q\bar{q}g_1 \cdots g_n$, there are $n!$ chain states $\{|f_i\rangle\}$, each connecting q with \bar{q} through n gluons in a specific order. A set of orthogonal states $\{|f_i^1\rangle\}$ is related to $\{|f_j\rangle\}$ by a linear transformation U_{ij}^1 , i.e. $|f_i^1\rangle = U_{ij}^1 |f_j\rangle$, where U_{ij}^1 is not a unitary matrix because of the non-orthogonality of the original states $\{|f_j\rangle\}$. U_{ij}^1 is not unique, so one can find many different ways of orthogonalization. Two different matrices U^1 and U^2 are associated with each other via a unitary matrix U^{12} , which guarantees the conservation of probability. This means that for two sets of orthogonal states, $|f_i^1\rangle$ and $|f_i^2\rangle$, which are both associated with the same set of non-orthogonal states $\{|f_j\rangle\}$, the following identity holds: $\sum_i |\langle f_i^1 | H_c \rangle|^2 = \sum_i |\langle f_i^2 | H_c \rangle|^2$.

Due the rapidly growing complexity of orthogonalization for large numbers of emitted gluons, let us only discuss the two simplest cases $e^+e^- \rightarrow q\bar{q}g_1g_2$ and $e^+e^- \rightarrow q\bar{q}g_1g_2g_3$.

For $e^+e^- \rightarrow q\bar{q}g_1g_2$, there are two chain states:

$$(|f_1\rangle, |f_2\rangle) \equiv \left(\frac{1}{3\sqrt{3}} |1_{q1}1_{12}1_{2\bar{q}}\rangle, \frac{1}{3\sqrt{3}} |1_{q2}1_{21}1_{1\bar{q}}\rangle \right)$$

where $|f_1\rangle$ and $|f_2\rangle$ are not orthogonal to each other: $\langle f_1 | f_2 \rangle = 1/N_c^2 = 1/9$. A straightforward way to construct orthogonal states is to linearly transform $|f_1\rangle$ and $|f_2\rangle$ into symmetric and anti-symmetric states respectively, i.e.

$$\begin{pmatrix} |f'_1\rangle \\ |f'_2\rangle \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{20}} & \frac{3}{\sqrt{20}} \\ \frac{3}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} |f_1\rangle \\ |f_2\rangle \end{pmatrix}, \quad (6)$$

where $|f'_1\rangle$ is the symmetric and $|f'_2\rangle$ the anti-symmetric state. Using $\langle f_1 | H_c \rangle = \frac{32}{9\sqrt{3}} D^{12} - \frac{4}{9\sqrt{3}} D^{21}$ and $\langle f_2 | H_c \rangle = -\frac{4}{9\sqrt{3}} D^{12} + \frac{32}{9\sqrt{3}} D^{21}$, we obtain the sum of squared projections as follows:

$$\begin{aligned} & |\langle f'_1 | H_c \rangle|^2 + |\langle f'_2 | H_c \rangle|^2 \\ &= \left(\frac{14^2}{27 \cdot 5} + 3 \right) (|D^{12}|^2 + |D^{21}|^2) + 2 \left(\frac{14^2}{27 \cdot 5} - 3 \right) \text{Re}(D^{12} \cdot D^{21*}) \\ &\approx 4.45 (|D^{12}|^2 + |D^{21}|^2) - 3.1 \text{Re}(D^{12} \cdot D^{21*}) \end{aligned} \quad (7)$$

The invariant amplitude for $e^+e^- \rightarrow q\bar{q}g_1g_2$ is: $M_{ab}^{\alpha_1\alpha_2} = (T^{\alpha_1}T^{\alpha_2})_{ab}D^{12} + (T^{\alpha_2}T^{\alpha_1})_{ab}D^{21}$. The total cross section of $e^+e^- \rightarrow q\bar{q}g_1g_2$ at the tree level is then:

$$\begin{aligned} \sigma_{tree}(e^+e^- \rightarrow q\bar{q}g_1g_2) &= \int d\Omega M_{ab}^{\alpha_1\alpha_2} (M_{ab}^{\alpha_1\alpha_2})^* \\ &= \int d\Omega \left[\frac{16}{3} (|D^{12}|^2 + |D^{21}|^2) - \frac{4}{3} \text{Re}(D^{12} \cdot D^{21*}) \right] \end{aligned} \quad (8)$$

The functions D^{12} and D^{21} correspond to different kinematical distributions, which in general have a rather limited overlap. Therefore the kinematic interference term proportional to $\text{Re}(D^{12} \cdot D^{21*})$ is suppressed. This kinematic interference term can be calculated in 2nd

order perturbation theory (we will come to this later). The result depends on the kinematical configuration, e. g. expressed by a y -cut for the definition of the 4-jet events. If we neglect kinematic interference terms in (7) and (8), we obtain:

$$P(e^+e^- \rightarrow q\bar{q}g_1g_2 \rightarrow SCS) = \frac{\sigma(e^+e^- \rightarrow q\bar{q}g_1g_2 \rightarrow SCS)}{\sigma_{tree}(e^+e^- \rightarrow q\bar{q}g_1g_2)} \simeq 83\% \quad (9)$$

where we mention again that SCS is the abbreviation for singlet chain states.

In order to evaluate the approximation of dropping kinematic interference terms, let us calculate the fraction $P(e^+e^- \rightarrow q\bar{q}g_1g_2 \rightarrow SCS)$ exactly. For a higher order process with more gluons produced, the calculation is too complicated. We know there are 8 lowest order Feynman diagrams in $e^+e^- \rightarrow q\bar{q}g_1g_2$, containing 2 diagrams with a tri-gluon vertex. In the calculation we choose the Feynman gauge, and we replace the polarization sum $\sum_{\lambda=1,2} \epsilon_\lambda^\mu(p) \epsilon_\lambda^\nu(p)$ with $-g^{\mu\nu}$, where the sum is taken over two transverse polarizations $\lambda = 1, 2$. However, $-g^{\mu\nu}$ equals to the sum over all four polarizations, including two unphysical ones. To cancel unphysical polarization states and guarantee unitarity, we should introduce two ghost diagrams. Of course, we may work in the physical gauge, where there is no ghost, and directly use physical polarizations for the gluons, but the calculation in the Feynman gauge is much simpler; see Ref. [7] for details. Thus, including two ghost diagrams, we have 10 diagrams altogether. The fact that ghost diagrams don't interfere with the 8 gluon diagrams make them easier to deal with. When we calculate the square of the amplitude $M = M_1 + M_2 + \dots + M_{10}$, we know that a non-interference term $|M_i|^2$ has a color factor $16/3$, while an interference term $M_i M_j^*$ has a color factor $-2/3$. For a tri-gluon and ghost-ghost-gluon vertex, we make substitution $T^a T^b - T^b T^a = i f^{abc} T^c$ to get the functions D^{12} and D^{21} . According to Eq.(7) and (8), we have:

$$P(e^+e^- \rightarrow q\bar{q}g_1g_2 \rightarrow SCS) = \frac{\int d\Omega [4.45(|D^{12}|^2 + |D^{21}|^2) - 3.1 \text{Re}(D^{12} \cdot D^{21*})]}{\int d\Omega [\frac{16}{3}(|D^{12}|^2 + |D^{21}|^2) - \frac{4}{3} \text{Re}(D^{12} \cdot D^{21*})]} \quad (10)$$

We use a Monte Carlo method to do the phase space integration and evaluate this ratio. The result is shown in Fig.4 as a function of a cutoff $y_{cut} = (p_i + p_j) \cdot (p_i + p_j)/s$ where p_i is the 4-momentum of parton i and \sqrt{s} is the center-of-mass energy of the e^+e^- collision, which we set to 91GeV. In the figure, we see that the rate decreases slowly, from 0.72 to 0.67, as y_{cut} varies from 10^{-4} to 10^{-2} . These values are smaller than the rate 0.83 obtained by neglecting the kinematic interference contribution, but the difference is not large. This implies that kinematic interference terms are less important than the non-interference terms.

For the $q\bar{q}g_1g_2g_3$ system, there are 6 singlet chain states ($|f_i\rangle, i = 1, 2 \dots 6$) which connect q to \bar{q} via three ordered gluons: 123, 231, 312, 213, 132, 321, respectively. These states are not orthogonal to each other. The inner product $\langle f_i | f_j \rangle$ is 1 for $i = j$ and $1/9$ for $i \neq j$. Our goal is to find 6 orthogonal states from them. The new states are denoted ($|f'_i\rangle, i = 1, 2 \dots 6$). Making use of the symmetric and approximately orthogonal properties of $\langle f_i | f_j \rangle$, we find one orthogonal set as follows:

$$|f'_i\rangle = (1 + \sigma) |f_i\rangle + \epsilon \sum_{j \neq i} |f_j\rangle, \quad \text{for } i = 1, \dots, 6 \quad (11)$$

where $\sigma \approx 0.021$, $\epsilon \approx -0.037$. According to arguments given for $e^+e^- \rightarrow q\bar{q}g_1g_2$, we may find a different set of orthogonal states which is related to $|f'_i\rangle$ via a unitary transformation, and we know that either set is equivalent.

Now we try to calculate the total fraction for chain states $|f'_i\rangle$ by projecting them on $|H_c\rangle$. Since each state is orthogonal to any other, we can sum up all squared projections:

$$\begin{aligned} \sum_{i=1}^6 |\langle f'_i | H_c \rangle|^2 &= 5.47 \{ |D^{123}|^2 + |D^{231}|^2 + |D^{312}|^2 \\ &+ |D^{213}|^2 + |D^{132}|^2 + |D^{321}|^2 \} \\ &+ \text{kinematic interference terms} \end{aligned} \quad (12)$$

where kinematic interference terms refer to interference terms between two different D . Thus we obtain for chain states in $e^+e^- \rightarrow q\bar{q}g_1g_2g_3$:

$$\sigma(e^+e^- \rightarrow SCS) = \int d\Omega \sum_{i=1}^6 |\langle f'_i | H_c \rangle|^2 \quad (13)$$

In order to estimate the fraction of chain states, we need to know the total cross section for $e^+e^- \rightarrow q\bar{q}g_1g_2g_3$ at the tree level:

$$\begin{aligned} \sigma_{tree}(e^+e^- \rightarrow q\bar{q}g_1g_2g_3) &= \int d\Omega \cdot M_{ab}^{\alpha_1\alpha_2\alpha_3} \cdot (M_{ab}^{\alpha_1\alpha_2\alpha_3})^* \\ &= \int d\Omega (\sum_P (T^{\alpha_{P(1)}} T^{\alpha_{P(2)}} T^{\alpha_{P(3)}})_{ab} D^P) (\sum_{P'} (T^{\alpha_{P'(1)}} T^{\alpha_{P'(2)}} T^{\alpha_{P'(3)}})_{ab} D^{P'})^* \end{aligned} \quad (14)$$

Here we resume the convention that a repetition of indices stands for summation. Expanding (14) gives:

$$\begin{aligned} \sigma_{tree}(e^+e^- \rightarrow q\bar{q}g_1g_2g_3) &= \int d\Omega \cdot \{ \frac{1}{2} \frac{(N_c^2-1)^3}{(2N_c)^2} [|D^{123}|^2 + |D^{231}|^2 + |D^{312}|^2 \\ &+ |D^{213}|^2 + |D^{132}|^2 + |D^{321}|^2] \\ &+ \text{kinematic interference terms} \} \end{aligned} \quad (15)$$

where kinematic interference terms are suppressed by powers of $1/N_c$. Thus, to leading order in N_c , we can give an instant estimate for the fraction from (13) and (15) without carrying out phase space integrals:

$$P(e^+e^- \rightarrow SCS) = \frac{\sigma(e^+e^- \rightarrow SCS)}{\sigma_{tree}(e^+e^- \rightarrow q\bar{q}g_1g_2g_3)} = \frac{5.47}{\frac{1}{2} \frac{(N_c^2-1)^3}{(2N_c)^2}} \sim 77\%$$

In this section, we have discussed the orthogonalization for chain states and estimated their fraction for $e^+e^- \rightarrow q\bar{q}g_1g_2$ and $e^+e^- \rightarrow q\bar{q}g_1g_2g_3$ by neglecting kinematic interference and then keeping only the interference due to the finite number of colors, N_c . There are many ways of constructing orthogonal states from the original non-orthogonal ones. Different ways lead to different orthogonal states. They are, however, equivalent for calculations of probabilities. Normally we can make use of the fact that the original states are approximately orthogonal up to $O(1/N_c^2)$. Thus one can find a set of orthogonal states, which are slightly different from the original ones, i.e. the transformation matrix is close to the unit matrix. The other straightforward orthogonalization recipe we give in this section, is to symmetrize and anti-symmetrize non-orthogonal states. We know that kinematic interference terms are all suppressed by $O(1/N_c)$ with respect to the non-interference terms. If we neglect all kinematic interference terms and then keep only the color interference brought by finite N_c , the total fraction is 83% for $e^+e^- \rightarrow q\bar{q}g_1g_2$ and 77% for $e^+e^- \rightarrow q\bar{q}g_1g_2g_3$. For the sake

of estimating the magnitude of the kinematic interference, we give the numerical result for the fraction $P(e^+e^- \rightarrow q\bar{q}g_1g_2 \rightarrow SCS)$ with the kinematic interference taken into account. The result is shown in Fig.4 as a function of y_{cut} . We see that the rate decreases slowly, from 0.72 to 0.67, as y_{cut} varies from 10^{-4} to 10^{-2} . These values are smaller than the rate 0.83 obtained by neglecting kinematic interference terms, but the difference is not large. This implies that kinematic interference terms are less important than non-interference ones, though not negligibly small.

V. PROPERTIES OF SINGLET CHAIN STATES FOR $Q\bar{Q} + NG$ IN THE LARGE N_C LIMIT

In this section, we will study the properties of the chain states for $q\bar{q} + ng$ in the large N_c limit and obtain their fraction to $O(1/N_c^2)$.

The projection of a chain state $|f\rangle$ on $|H_c\rangle$ is:

$$\langle f | H_c \rangle = \sum_P (1/\sqrt{2})^n D^P \langle f | \text{Tr}(QG_{P(1)}G_{P(2)} \cdots G_{P(n)}) \rangle$$

where H_c is given by (1). Without loss of generality, we choose

$$|f\rangle = N_c^{-(n+1)/2} |1_{q1}1_{12}1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}}\rangle$$

The projection of any other chain state can be obtained by permuting gluon labels. For convenience, we will ignore the normalization factor $N_c^{-(n+1)/2}$ and denote $|f\rangle$ as equivalent to $|1_{q1}1_{12}1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}}\rangle$ in intermediate steps of the calculation. We will put the normalization factor back in the final results.

The order of the n gluons in $|f\rangle$ is $(1, 2, 3, \cdots, n)$. In $|H_c\rangle$, there is also a term $|\text{Tr}(QG_1G_2 \cdots G_n)\rangle$ with gluon labels in the same order. Let us first calculate

$$\langle f | \text{Tr}(QG_1G_2 \cdots G_n) \rangle = N_c^{-(n+1)/2} \langle 1_{q1}1_{12}1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}} | \text{Tr}(QG_1G_2 \cdots G_n) \rangle \quad (16)$$

We can expand $|\text{Tr}(QG_1G_2 \cdots G_n)\rangle$ as follows:

$$\begin{aligned} |\text{Tr}(QG_1G_2 \cdots G_n)\rangle &= |\text{Tr}(QG'_1G'_2 \cdots G'_n)\rangle \\ &+ \sum_{k=1}^n \sum_{\{u_1, \dots, u_k\}} \left(-\frac{1}{N_c}\right)^k |1_{u_1}1_{u_2} \cdots 1_{u_k}\rangle \left| \text{Tr}(QG'_{v_1}G'_{v_2} \cdots G'_{v_{n-k}}) \right\rangle \end{aligned} \quad (17)$$

where $(v_1, v_2, \cdots, v_{n-k})$ is the supplementary set to (u_1, u_2, \cdots, u_k) in $(1, 2, 3, \cdots, n)$, and it satisfies $v_1 < v_2 < \cdots < v_{n-k}$, i.e. the relative order of these $n-k$ gluons in $(G'_{v_1}G'_{v_2} \cdots G'_{v_{n-k}})$ remains the same as in $(QG_1G_2 \cdots G_n)$; $\sum_{\{u_1, \dots, u_k\}}$ sums over all decompositions of $(1, 2, \cdots, n)$ into (u_1, u_2, \cdots, u_k) and $(v_1, v_2, \cdots, v_{n-k})$. Note that $|\text{Tr}(QG'_1G'_2 \cdots G'_n)\rangle$ is just $|1_{q1}1_{12}1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}}\rangle$. Thus $|\text{Tr}(QG_1G_2 \cdots G_n)\rangle$ can also be written as:

$$\begin{aligned} |\text{Tr}(QG_1G_2 \cdots G_n)\rangle &= |1_{q1}1_{12}1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}}\rangle \\ &+ \sum_{k=1}^n \sum_{\{u_1, \dots, u_k\}} \left(-\frac{1}{N_c}\right)^k |1_{u_1}1_{u_2} \cdots 1_{u_k}\rangle \left| 1_{qv_1}1_{v_1v_2}1_{v_2v_3} \cdots 1_{v_iv_{i+1}} \cdots 1_{v_{n-k}\bar{q}} \right\rangle \end{aligned} \quad (18)$$

We find immediately

$$\langle 1_{q1}1_{12}1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}} | \text{Tr}(QG'_1G'_2 \cdots G'_n) \rangle = N_c^{n+1}$$

Let us calculate

$$\langle 1_{q1}1_{12}1_{23}\cdots 1_{i,i+1}\cdots 1_{n\bar{q}} \mid 1_u1_{q1}1_{12}1_{23}\cdots 1_{u-2,u-1}1_{u-1,u+1}\cdots 1_{n\bar{q}} \rangle,$$

This is one of the terms in (18) with $k = 1$. Therefore we denote it $\langle f \mid k = 1 \text{ term} \rangle$ in short. We draw a diagram as shown in Fig.2 where corresponding to each 1_{st} , there is a line starting from position s in the lower row to position t in the upper row. Counting the number of closed paths, we obtain

$$\begin{aligned} & \langle 1_{q1}1_{12}1_{23}\cdots 1_{i,i+1}\cdots 1_{n\bar{q}} \mid 1_u1_{q1}1_{12}1_{23}\cdots 1_{u-2,u-1}1_{u-1,u+1}\cdots 1_{n\bar{q}} \rangle \\ &= N_c^{n+1-2+1} = N_c^n \end{aligned} \quad (19)$$

It is a little more complicated to calculate

$$\begin{aligned} & \langle f \mid k = 2 \text{ term} \rangle \\ &= \langle 1_{q1}1_{12}1_{23}\cdots 1_{i,i+1}\cdots 1_{n\bar{q}} \mid 1_{u1}1_{u2}(1_{qv1}1_{v1v2}1_{v2v3}\cdots 1_{v_iv_{i+1}}\cdots 1_{v_{n-k}\bar{q}}) \rangle \end{aligned} \quad (20)$$

There are two cases: one where u_1 and u_2 are adjacent to each other, the other where u_1 and u_2 are not adjacent. To see more clearly, we look at the two cases separately, and we show the corresponding diagrams in Fig.3. For the case when u_1 and u_2 are adjacent, we immediately have

$$\langle f \mid k = 2 \text{ term} \rangle = N_c^{n+1-3+1} = N_c^{n-1}$$

When u_1 and u_2 are not neighbors, we find

$$\langle f \mid k = 2 \text{ term} \rangle = N_c^{n+1-2\times 2+2} = N_c^{n-1}$$

Thus we obtain the same value in both cases.

As a matter of fact, for a general expression

$$\begin{aligned} & \langle f \mid k \text{ term} \rangle \\ & \equiv \langle 1_{q1}1_{12}1_{23}\cdots 1_{i,i+1}\cdots 1_{n\bar{q}} \mid (1_{u1}1_{u2}\cdots 1_{u_k})(1_{qv1}1_{v1v2}1_{v2v3}\cdots 1_{v_iv_{i+1}}\cdots 1_{v_{n-k}\bar{q}}) \rangle \end{aligned} \quad (21)$$

where $k = 3, 4, 5, \dots, n$, there is a unique value regardless of whether u_1, u_2, \dots, u_k or part of them are neighbors. Let us distinguish two cases with and without adjacent parton labels. If no label in u_1, u_2, \dots, u_k is adjacent to another, we have

$$\langle f \mid k \text{ term} \rangle = N_c^{n+1-2k+k} = N_c^{n-k+1}$$

where $n + 1 - 2k$ is the number of double-line loops and k is the number of closed paths involving u_1, u_2, \dots, u_k . If there are m labels, each of which is adjacent to at least one other, m can be grouped into l non-adjacent segments where labels belonging to the same segment are continuous, i.e. $m = \sum_{i=1}^l m_i$ where m_i is the number of labels in the i -th segment. Hence we get the result $\langle f \mid k \text{ term} \rangle = N_c^\epsilon$ where ϵ is

$$\begin{aligned} \epsilon &= [n + 1 - 2(k - m) - (m_1 + 1) - (m_2 + 1) - \cdots (m_l + 1)] \\ &+ [k - m] + l = n - k + 1 \end{aligned} \quad (22)$$

Here the first term is the number of double-line loops, the second term is the contribution from $k - m$ separated labels and the third term is from l continuous segments. Finally we have,

$$\begin{aligned}
& \langle 1_{q1} 1_{12} 1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}} \mid Tr(QG_1 G_2 \cdots G_n) \rangle \\
&= N_c^{n+1} + \sum_{k=1}^n C_n^k (-1)^k \frac{1}{N_c^k} N_c^{n-k+1} \\
&= N_c^{n+1} + \sum_{k=1}^n C_n^k (-1)^k N_c^{n-2k+1}
\end{aligned} \tag{23}$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ denotes the number of ways to pick k out of n labels.

Now we start calculating the general inner product $\langle f \mid Tr(QG_{P(1)} G_{P(2)} \cdots G_{P(n)}) \rangle$. First we focus on one of the simplest cases:

$$\langle f \mid Tr(QG_1 G_3 G_2 G_4 G_5 \cdots G_n) \rangle \tag{24}$$

where two adjacent gluon labels 2 and 3 are interchanged compared to (16). Similar to (17), we expand $\mid Tr(QG_1 G_3 G_2 G_4 G_5 \cdots G_n) \rangle$ as follows:

$$\begin{aligned}
& \mid Tr(QG_1 G_3 G_2 G_4 G_5 \cdots G_n) \rangle \\
&= \mid Tr(QG'_1 G'_3 G'_2 G'_4 G'_5 \cdots G'_n) \rangle \\
&+ \sum_{k=1}^n \sum_{\{u_1, \dots, u_k\}} \left(-\frac{1}{N_c}\right)^k \mid 1_{u_1} 1_{u_2} \cdots 1_{u_k} \rangle \mid Tr(QG'_{v_1} G'_{v_2} \cdots G'_{v_{n-k}}) \rangle
\end{aligned} \tag{25}$$

According to (4), the first term of (24) is:

$$\begin{aligned}
& \langle 1_{q1} 1_{12} 1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}} \mid 1_{q1} 1_{13} 1_{32} 1_{24} 1_{45} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}} \rangle \\
&= N_c^{n+1} / N_c^2 = N_c^{n-1}
\end{aligned}$$

Secondly we consider the following $k = 1$ terms:

$$(-1/N_c) \langle 1_{q1} 1_{12} 1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}} \mid 1_2 (1_{q1} 1_{13} 1_{34} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}}) \rangle \tag{26}$$

and

$$(-1/N_c) \langle 1_{q1} 1_{12} 1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}} \mid 1_3 (1_{q1} 1_{12} 1_{24} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}}) \rangle \tag{27}$$

where gluon 2 and 3 are picked out as singlet 1_2 and 1_3 respectively. Note that the only difference between $\mid Tr(QG_1 G_3 G_2 G_4 G_5 \cdots G_n) \rangle$ and $\mid Tr(QG_1 G_2 G_3 G_4 G_5 \cdots G_n) \rangle$ is that the order of gluon 2 and 3 is interchanged. Thus we also encounter (26) and (27) in calculating $\langle f \mid Tr(QG_1 G_2 \cdots G_n) \rangle$. From (19) we see that (26) and (27) give the same value $-N_c^{n-1}$. For other terms we have

$$(-1/N_c) \langle f \mid 1_u (1_{q1} 1_{13} 1_{32} 1_{24} 1_{45} \cdots 1_{u-2,u-1} 1_{u-1,u+1} \cdots 1_{n\bar{q}}) \rangle = -N_c^{n-3} \tag{28}$$

where $u \neq 2, 3$. We see that (28) is suppressed by an additional factor $1/N_c^2$ compared to (26) and (27). It is easy to show that the contribution from terms

$$\left(-\frac{1}{N_c}\right)^k \mid 1_{u_1} 1_{u_2} \cdots 1_{u_k} \rangle \mid Tr(QG'_{v_1} G'_{v_2} \cdots G'_{v_{n-k}}) \rangle$$

with $k > 1$ are suppressed at least by $1/N_c^2$ as compared to (28). Thus, up to the highest order, we have:

$$\langle f \mid Tr(QG_1 G_3 G_2 G_4 G_5 \cdots G_n) \rangle = -N_c^{n-1} + O(N_c^{n-3}) \tag{29}$$

There are $(n - 1)$ trace terms in $|H_c\rangle$, which differ from $|f\rangle$ only in the order of two adjacent gluons. Their inner products with $|f\rangle$ are given by (29). There is a set of trace terms for which the labels of two non-adjacent gluons are interchanged compared to $|f\rangle$. We can in the same way prove that the projection of $|f\rangle$ on these trace terms gives $N_c^{n-1} + O(N_c^{n-3})$, which is in the same magnitude as (29) but opposite in sign. These are next-to-leading order terms ($\sim N_c^{n-1}$) compared to the leading term of the order N_c^{n+1} in (23). Remaining terms have inner products which are suppressed by more powers of $1/N_c$.

Corresponding to a chain state $|f\rangle$, we can classify all trace terms in $|H_c\rangle$ into three groups. One is the leading term where the gluon order is the same as that in $|f\rangle$. The second group are the next-to-leading terms, where the order of two gluons is interchanged relative to $|f\rangle$. This group can be further classified into two subgroups according to whether the order of two adjacent or non-adjacent gluons is interchanged respectively. The third group contains higher order terms, in which the order of the gluons differs even more from $|f\rangle$. The leading term is denoted as $L(f)$, the next-to-leading terms as $NL(f)$ with $NL_1(f)$ and $NL_2(f)$ for the adjacent and non-adjacent case respectively, and higher terms as $H(f)$. Hence, for the state

$$|f\rangle = N_c^{-(n+1)/2} |1_{q1}1_{12}1_{23}\cdots 1_{i,i+1}\cdots 1_{n\bar{q}}\rangle$$

we find that, up to next-to-leading order, the projection on $|H_c\rangle$ is given by

$$\begin{aligned} \langle f | H_c \rangle &= \sum_P (1/\sqrt{2})^n D^P \langle f | \text{Tr}(QG_{P(1)}G_{P(2)}\cdots G_{P(n)}) \rangle \\ &\approx (1/\sqrt{2})^n \cdot N_c^{(n+1)/2} \cdot \left[\left(1 - \frac{n}{N_c^2}\right) D^{L(f)} - \frac{1}{N_c^2} \sum_{P \in NL_1(f)} D^P + \frac{1}{N_c^2} \sum_{P \in NL_2(f)} D^P \right] \end{aligned} \quad (30)$$

We know from the previous section that there are $n!$ singlet chain states which are denoted $\{|f_i\rangle, i = 1, 2, \dots, n!\}$. They are not orthogonal to each other. The largest inner product of two states is $1/N_c^2$. Suppose we find a set of orthogonal states $\{|f'_i\rangle\}$ from $\{|f_i\rangle\}$. Up to next-to-leading order, we assume $\{|f'_i\rangle\}$ can be written in this form:

$$|f'_i\rangle \approx \left(1 + \frac{C_1}{N_c^2}\right) |f_i\rangle - \frac{C_2}{N_c^2} \sum_{j \in NL'(f_i)} |f_j\rangle \quad (31)$$

where $NL'(f_i)$ refers to the set of chain states which contribute to $|f'_i\rangle$ in next-to-leading order; C_1 and C_2 are constants of order 1. One can verify that the set NL is included in NL' . The reason is that for a chain state the colors of the quark-antiquark pair play an equal role as those of gluons, while for a trace term in $|H_c\rangle$ it emerges as a color nonet which is different from the color octets of gluons. To see it more clearly, we take as an example two permutations (01234) and (02341). Obviously for chain states $\langle 01234 | 02341 \rangle = (1/N_c^2) \langle 01234 | 01234 \rangle$, hence the orders (01234) and (02341) belong to the same NL' set. Here we use a simplified notation for the chain state, e.g. $|01234\rangle \equiv |1_{01}1_{12}1_{23}1_{34}1_{40}\rangle$. It is easy to understand this because we can write the second state in the form $|10234\rangle$, which is different from the state $|01234\rangle$ in two labels 0 and 1. We can verify that $\langle 01234 | \text{Tr}(02341) \rangle$ is suppressed by $O(1/N_c^4)$ relative to $\langle 01234 | \text{Tr}(01234) \rangle$, i.e. $|\text{Tr}(02341)\rangle$ does not belong to $NL(01234)$. Here we use a simplified notation for the chain state, e.g. $\text{Tr}(01234) \equiv \text{Tr}(QG_1G_2G_3G_4)$. Another different feature of NL' compared to NL is that the inner product of any two different states, where only two gluon labels are interchanged, is always suppressed by $1/N_c^2$ relative to the inner product of themselves, regardless of whether they

belong to the adjacent or non-adjacent case. Hence NL' can be written in this form: $NL' = NL_1 + NL_2 + \overline{NL}$. Up to $O(1/N_c^2)$ we have

$$\begin{aligned}
\langle f'_i | H_c \rangle &= (1 + \frac{C_1}{N_c^2}) \langle f_i | H_c \rangle - \frac{C_2}{N_c^2} \sum_{j \in NL(f_i)} \langle f_j | H_c \rangle \\
&= 2^{-n/2} N_c^{(n+1)/2} \left[\left(1 + \frac{C_1}{N_c^2}\right) \left(1 - \frac{n}{N_c^2}\right) D^{L(f_i)} - \frac{1}{N_c^2} \sum_{P \in NL_1(f_i)} D^P \right. \\
&\quad \left. + \frac{1}{N_c^2} \sum_{P \in NL_2(f_i)} D^P - \frac{C_2}{N_c^2} \sum_{j \in NL(f_i)} D^{L(f_j)} \right] \\
&= 2^{-n/2} N_c^{(n+1)/2} \left[\left(1 + \frac{C_1-n}{N_c^2}\right) D^{L(f_i)} - \frac{C_2+1}{N_c^2} \sum_{P \in NL_1(f_i)} D^P \right. \\
&\quad \left. - \frac{C_2-1}{N_c^2} \sum_{P \in NL_2(f_i)} D^P - \frac{C_2}{N_c^2} \sum_{P \in \overline{NL}(f_i)} D^P \right]
\end{aligned} \tag{32}$$

and the projection square has the following form:

$$\begin{aligned}
|\langle f'_i | H_c \rangle|^2 &= 2^{-n} N_c^{n+1} \left[\left(1 + \frac{2(C_1-n)}{N_c^2}\right) |D^{L(f_i)}|^2 - \frac{2(C_2+1)}{N_c^2} \sum_{P \in NL_1(f_i)} \text{Re}(D^{L(f_i)} \cdot D^{P*}) \right. \\
&\quad \left. - \frac{2(C_2-1)}{N_c^2} \sum_{P \in NL_2(f_i)} \text{Re}(D^{L(f_i)} \cdot D^{P*}) - \frac{2C_2}{N_c^2} \sum_{P \in \overline{NL}(f_i)} \text{Re}(D^{L(f_i)} \cdot D^{P*}) \right]
\end{aligned} \tag{33}$$

where we only keep the next-to-leading order. The total sum is

$$\begin{aligned}
\sum_{i=1}^{n!} |\langle f'_i | H_c \rangle|^2 &= 2^{-n} N_c^{n+1} \left[\left(1 + \frac{2(C_1-n)}{N_c^2}\right) \sum_P |D^P|^2 - \frac{4(C_2+1)}{N_c^2} \sum_{\{P, P'\} \in NL_1} \text{Re}(D^P \cdot D^{P'*}) \right. \\
&\quad \left. - \frac{4(C_2-1)}{N_c^2} \sum_{\{P, P'\} \in NL_2} \text{Re}(D^P \cdot D^{P'*}) - \frac{4C_2}{N_c^2} \sum_{P \in \overline{NL}(f_i)} \text{Re}(D^{L(f_i)} \cdot D^{P*}) \right]
\end{aligned} \tag{34}$$

where $\{P, P'\} \in NL_1$ means that the orders P and P' are different in only two neighboring gluon labels, $\{P, P'\} \in NL_2$ means that they are different in two non-adjacent gluon labels, and $\{P, P'\} \in \overline{NL}$ means $\{P, P'\} \in NL'$ but not included in $NL_1 + NL_2$.

Now we start calculating $\sigma_{tree}(e^+e^- \rightarrow q\bar{q} + ng)$ in the large N_c limit. Recall that the ordinary matrix element is given by

$$M_{ab}^{\alpha_1 \alpha_2 \dots \alpha_n} = \sum_P (T^{\alpha_{P(1)}} T^{\alpha_{P(2)}} \dots T^{\alpha_{P(n)}})_{ab} D^P$$

The total cross section is then:

$$\begin{aligned}
\sigma_{tree}(e^+e^- \rightarrow q\bar{q} + ng) &= \sum_{a,b,\alpha_1,\alpha_2,\dots,\alpha_n} \int d\Omega |M_{ab}^{\alpha_1 \alpha_2 \dots \alpha_n}|^2 \\
&= \sum_{a,b,\alpha_1,\alpha_2,\dots,\alpha_n} \int d\Omega \left| \sum_P (T^{\alpha_{P(1)}} T^{\alpha_{P(2)}} \dots T^{\alpha_{P(n)}})_{ab} D^P \right|^2
\end{aligned} \tag{35}$$

In evaluating (35), we mainly encounter two types of traces of Gell-Mann matrices

$$\begin{aligned}
&\text{Tr}(T^{\alpha_1} T^{\alpha_2} \dots T^{\alpha_n} T^{\alpha_n} \dots T^{\alpha_2} T^{\alpha_1}) \\
&= N_c \cdot C_F^n = N_c \left(\frac{N_c^2 - 1}{2N_c} \right)^n \\
&\approx (1/2)^n \cdot N_c^{(n+1)} \left(1 - \frac{n}{N_c^2} \right)
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
\text{Tr}(A_1 T^\alpha A_2 T^\alpha A_3) &= \frac{1}{2} \text{Tr}(A_2) \text{Tr}(A_1 A_3) - \frac{1}{2N_c} \text{Tr}(A_1 A_2 A_3) \\
\text{Tr}(A_1 T^\alpha) \text{Tr}(A_2 T^\alpha) &= \frac{1}{2} \text{Tr}(A_1 A_2) - \frac{1}{2N_c} \text{Tr}(A_1) \text{Tr}(A_2)
\end{aligned} \tag{37}$$

where A_1, A_2 and A_3 are chains of products of Gell-Mann matrices. The result in (36) is just the color factor associated with the terms $|D^P|^2$, while (37) gives the color factor for the

kinematic interference terms $D^P \cdot D^{P'}$, where P and P' denote two different permutations. In the large N_c limit the factor (36) is the leading one. The next-to-leading contribution comes from $D^P \cdot D^{P'}$ terms where P and P' are different in only two gluon labels. Also the next-to-leading contributions can be classified into two different cases, one is that $\{P, P'\} \in NL_1$, i.e. P and P' are different in two neighboring gluon labels, the other is that $\{P, P'\} \in NL_2$, i.e. they are different in two non-adjacent gluon labels. For $\{P, P'\} \in NL_1$, the color factor is $-\frac{1}{2N_c} \cdot N_c \cdot C_F^{n-1} \approx -\frac{1}{2^n} N_c^{n-1}$, while for $\{P, P'\} \in NL_2$, the color factor is $\frac{1}{2^n} N_c^{n-1}$ in the large N_c limit. After keeping terms up to next-to-leading order we have

$$\begin{aligned} \sigma_{tree}(e^+e^- \rightarrow q\bar{q} + ng) \\ = (1/2)^n \cdot N_c^{(n+1)} \cdot \int d\Omega [(1 - \frac{n}{N_c^2}) \sum_P |D^P|^2 - \frac{2}{N_c^2} \sum_{\{P, P'\} \in NL_1} \text{Re}(D^P \cdot D^{P'*}) \\ + \frac{2}{N_c^2} \sum_{\{P, P'\} \in NL_2} \text{Re}(D^P \cdot D^{P'*})] \end{aligned} \quad (38)$$

We write the fraction of singlet chain states as:

$$\begin{aligned} P(e^+e^- \rightarrow q\bar{q} + ng \rightarrow SCS) \\ = (\int d\Omega \sum_{i=1}^{n!} |\langle f'_i | H_c \rangle|^2) / \sigma_{tree}(e^+e^- \rightarrow q\bar{q} + ng) \end{aligned} \quad (39)$$

where $\sum_{i=1}^{n!} |\langle f'_i | H_c \rangle|^2$ is given by (34). According to Eq.(34, 38, 39) and we reach our final result up to next-to-leading order

$$\begin{aligned} P(e^+e^- \rightarrow q\bar{q} + ng \rightarrow SCS) \\ = 1 + \frac{1}{N_c^2} [(2C_1 - n) - (4C_2 + 2)T_1 - (4C_2 - 2)T_2 - 4C_2T_3] \end{aligned} \quad (40)$$

where T_1 and T_2 are defined by:

$$\begin{aligned} T_1 &= \frac{\int d\Omega \sum_{\{P, P'\} \in NL_1} \text{Re}(D^P \cdot D^{P'*})}{\int d\Omega \sum_P |D^P|^2} \\ T_2 &= \frac{\int d\Omega \sum_{\{P, P'\} \in NL_2} \text{Re}(D^P \cdot D^{P'*})}{\int d\Omega \sum_P |D^P|^2} \\ T_3 &= \frac{\int d\Omega \sum_{\{P, P'\} \in \overline{NL}} \text{Re}(D^P \cdot D^{P'*})}{\int d\Omega \sum_P |D^P|^2} \end{aligned} \quad (41)$$

The result in Eq.(40) gives the fraction of events, for which the colors of the gluons correspond to a single chain from the quark to the antiquark. We expect that these states hadronize producing a corresponding chain of hadrons. The remaining events correspond to more complicated color structures, where one gluon can be connected to more than two other gluons, as indicated in Fig. 2 and 3. Some dynamical feature of the confining mechanism may imply that also these states result in string-like hadronic final states, but it is also conceivable that these parton states can produce more complex hadron configurations.

VI. H_C IN SOFT GLUON BREMSSTRAHLUNG AND ITS RELATION TO DIPOLE CASCADE MODEL

We have not yet touched the momentum function D in H_c so far. The number and complexity of Feynman diagrams increases drastically with growing number of emitted gluons

in normal situations. However, in the case of soft gluon bremsstrahlung, the D function has a simple and recursive structure. The method we use to derive the D function or H_c in this section is called soft gluon insertion technique [8]. By recursively adding a softer (with lower energy) gluon in multigluon emissions, the distribution for each new emission is approximately determined by an eikonal current stemming from all the harder gluons. The result factorizes between the emissions and is equivalent to classical bremsstrahlung under certain angular ordering conditions.

Assume that in e^+e^- annihilation, $e^+e^- \rightarrow q\bar{q}g_1g_2\cdots g_n$, n gluons are all soft ones and their energies/momenta are strongly ordered:

$$E_p \sim E_{p'} \gg E_{k_1} \gg E_{k_2} \cdots \gg E_{k_n} \quad (42)$$

where $p, p', k_1, k_2, \dots, k_n$ are 4-momenta of $q, \bar{q}, g_1, g_2, \dots, g_n$ respectively. When the hardest gluon g_1 is emitted, it has two legs to attach to, one is the quark's momentum and the other is the anti-quark's. They give rise to the amplitude:

$$M_{ab}^{\alpha_1} \sim g_s \varepsilon_{\mu_1} J_{ab\alpha_1}^{\mu_1} = g_s \left(\frac{p^{\mu_1}}{p \cdot k_1} - \frac{p'^{\mu_1}}{p' \cdot k_1} \right) T_{ab}^{\alpha_1} \varepsilon_{\mu_1} \equiv g_s J^{\mu_1}(k_1; p, p') T_{ab}^{\alpha_1} \varepsilon_{\mu_1} \quad (43)$$

where ε denotes the gluon's polarization 4-vector and g_s is the strong coupling constant.

When the second hardest gluon g_2 is emitted, it has three legs to attach to: q, \bar{q} and g_1 . The amplitude is:

$$\begin{aligned} M_{ab}^{\alpha_1\alpha_2} &\sim g_s^2 \varepsilon_{\mu_1} \varepsilon_{\mu_2} J_{ab\alpha_1\alpha_2}^{\mu_1\mu_2} \\ &= g_s^2 \varepsilon_{\mu_1} \varepsilon_{\mu_2} J^{\mu_1}(k_1; p, p') \left[\frac{p^{\mu_2}}{p \cdot k_2} (T^{\alpha_2} T^{\alpha_1})_{ab} - \frac{p'^{\mu_2}}{p' \cdot k_2} (T^{\alpha_1} T^{\alpha_2})_{ab} + \frac{k_1^{\mu_2}}{k_1 \cdot k_2} T_{A\alpha_1\beta}^{\alpha_2} T_{ab}^{\beta} \right] \\ &= g_s^2 \varepsilon_{\mu_1} \varepsilon_{\mu_2} [J^{\mu_1}(k_1; p, p') J^{\mu_2}(k_2; p, k_1) (T^{\alpha_2} T^{\alpha_1})_{ab} \\ &\quad + J^{\mu_1}(k_1; p, p') J^{\mu_2}(k_2; k_1, p') (T^{\alpha_1} T^{\alpha_2})_{ab}] \end{aligned} \quad (44)$$

where $T_{A\alpha_1\beta}^{\alpha_2} = if_{\alpha_1\alpha_2\beta}$ is the generator of the adjoint representation of $SU(3)$, and the rule $if_{\alpha_1\alpha_2\beta} T^{\alpha_1} T^{\alpha_2} = T^{\alpha_1} T^{\alpha_2} - T^{\alpha_2} T^{\alpha_1}$ has been used.

In the case of n -gluon emission, we can prove in the same way that the amplitude and its corresponding $|H_c\rangle$ can be written in this form:

$$\begin{aligned} M_{ab}^{\alpha_1\alpha_2\cdots\alpha_n} &= \sum_P (T^{\alpha_{P(1)}} T^{\alpha_{P(2)}} \cdots T^{\alpha_{P(n)}})_{ab} D^P \\ &\sim g_s^n \varepsilon_{\mu_1} \varepsilon_{\mu_2} \cdots \varepsilon_{\mu_n} [\sum_P (T^{\alpha_{P(1)}} T^{\alpha_{P(2)}} \cdots T^{\alpha_{P(n)}})_{ab} \\ &\quad \cdot J^{\mu_1}(k_1; p, p') J^{\mu_2}(k_2; k_{2h}, k_{2e}) \cdots J^{\mu_n}(k_n; k_{nh}, k_{ne})] \\ |H_c\rangle &\sim g_s^n \varepsilon_{\mu_1} \varepsilon_{\mu_2} \cdots \varepsilon_{\mu_n} \sum_P (1/\sqrt{2})^n |Tr(QG_{P(1)}G_{P(2)}\cdots G_{P(n)})\rangle \\ &\quad \cdot J^{\mu_1}(k_1; p, p') J^{\mu_2}(k_2; k_{2h}, k_{2e}) \cdots J^{\mu_n}(k_n; k_{nh}, k_{ne}) \end{aligned} \quad (45)$$

where subscripts ih and ie (where $i = 2, \dots, n$) are determined by the following procedure: in the sequence $(0, P(1), P(2), \dots, P(n), 0)$ (where we imply $0 \equiv q$ at the head and $0 \equiv \bar{q}$ at the end), find the position of i , take away all greater numbers in the sequence, the left-nearest neighbor to i is ih and its right-nearest neighbor is ie . Having this complete form of $|H_c\rangle$, we can calculate the fraction of any color state by projecting the state onto it. Here we are only interested in what happens to SCS in the large N_c limit. We consider a chain state $|f_P\rangle$ which corresponds to a specific order of gluons: $(P(1), P(2), \dots, P(n))$ where P denotes the permutation of $(1, 2, \dots, n)$. According to the former section, in the leading

order, the inner product $\langle f_P | H_c \rangle$ only picks up the term with the same order of gluons in $|H_c\rangle$:

$$\langle f_P | H_c \rangle \simeq g_s^n \varepsilon_{\mu_1} \varepsilon_{\mu_2} \cdots \varepsilon_{\mu_n} J^{\mu_1}(k_1; p, p') J^{\mu_2}(k_2; k_{2h}, k_{2e}) \cdots J^{\mu_n}(k_n; k_{nh}, k_{ne}) \cdot (1/\sqrt{2})^n \langle f_P | \text{Tr}(Q G_{P(1)} G_{P(2)} \cdots G_{P(n)}) \rangle \quad (46)$$

According to (30), the projection square is:

$$|\langle f_P | H_c \rangle|^2 = N_c^{n+1} g_s^{2n} (p, p')_{k_1} (k_{2h}, k_{2e})_{k_2} \cdots (k_{nh}, k_{ne})_{k_n} \quad (47)$$

where the antenna term is defined by:

$$(p_1, p_2)_k \equiv \frac{p_1 \cdot p_2}{(k \cdot p_1)(k \cdot p_2)} \quad (48)$$

Hence we see that each gluon g_i is associated with two harder gluons g_{ih} and g_{ie} which are nearest to its position in the sequence.

As an example, we look at the case:

$$|f\rangle = N_c^{-(n+1)/2} |1_{q1} 1_{12} 1_{23} \cdots 1_{ii+1} \cdots 1_{n\bar{q}}\rangle \quad (49)$$

with gluons' order $(1, 2, \cdots, n)$. The projection square is:

$$|\langle f | H_c \rangle|^2 = N_c^{n+1} g_s^{2n} (p, p')_{k_1} (k_1, p')_{k_2} (k_2, p')_{k_3} \cdots (k_{n-1}, p')_{k_n} \quad (50)$$

The cross section of $e^+e^- \rightarrow q\bar{q} + ng \rightarrow |f\rangle$ is:

$$d\sigma(e^+e^- \rightarrow |f\rangle) = N_c^{n+1} g_s^{2n} \frac{d^3 p}{(2\pi)^3 2E} \frac{d^3 p'}{(2\pi)^3 2E'} \prod_{i=1}^n \frac{d^3 k_i}{(2\pi)^3 2E_i} (p, p')_{k_1} (k_1, p')_{k_2} (k_2, p')_{k_3} \cdots (k_{n-1}, p')_{k_n} (2\pi)^4 \delta^{(4)}(P_{e^+} + P_{e^-} - [p + p' + \sum k_i]) \quad (51)$$

In the approximation of soft gluon bremsstrahlung, the above cross section can be written as:

$$\begin{aligned} d\sigma(e^+e^- \rightarrow |f\rangle) &= d\sigma(e^+e^- \rightarrow q\bar{q}) \prod_{i=1}^n \frac{N_c}{2} \alpha_s \frac{d^3 k_i}{4\pi^2 4E_i} \frac{k_{i-1} \cdot p'}{(k_i \cdot k_{i-1})(k_i \cdot p')} \\ &= d\sigma(e^+e^- \rightarrow q\bar{q}) \prod_{i=1}^n \frac{N_c}{2\pi} \alpha_s \frac{d(k_i \cdot k_{i-1})}{(k_i \cdot k_{i-1})} \frac{d(k_i \cdot p')}{(k_i \cdot p')} \end{aligned} \quad (52)$$

where the strong coupling constant $\alpha_s \equiv \frac{g_s^2}{4\pi}$, and we use the notation $k_0 \equiv p$ for convenience. Define two new variables for the gluon g_i which are called the generalized rapidity and the transverse momentum:

$$y(g_{i-1}\bar{q} \rightarrow g_i) = \frac{1}{2} \ln \left(\frac{k_i \cdot k_{i-1}}{k_i \cdot p'} \right), \quad k_T^2(g_{i-1}\bar{q} \rightarrow g_i) = \frac{(k_i \cdot k_{i-1})(k_i \cdot p')}{k_{i-1} \cdot p'} \quad (53)$$

We can rewrite Eq. (52) in terms of y and p_T :

$$d\sigma(e^+e^- \rightarrow |f\rangle) = d\sigma(e^+e^- \rightarrow q\bar{q}) \prod_{i=1}^n d\sigma(g_{i-1}\bar{q} \rightarrow g_i) \quad (54)$$

where we define the emission rate for g_i :

$$d\sigma(g_{i-1}\bar{q} \rightarrow g_i) = \frac{N_c}{2\pi} \alpha_s dy(g_{i-1}\bar{q} \rightarrow g_i) \frac{dk_T^2(g_{i-1}\bar{q} \rightarrow g_i)}{k_T^2(g_{i-1}\bar{q} \rightarrow g_i)} \quad (55)$$

Eq. (54) is the result for one chain state with a specific order. If we take into account all $n!$ states, Eq.(54) becomes:

$$\begin{aligned} d\sigma(e^+e^- \rightarrow \sum_{j=1}^{n!} |f_j\rangle) &= d\sigma(e^+e^- \rightarrow q\bar{q}) d\sigma(q\bar{q} \rightarrow g_1) \\ &\cdot [d\sigma(qg_1 \rightarrow g_2) d\sigma(qg_2g_1\bar{q} \rightarrow g_3 \cdots g_n) \\ &+ d\sigma(g_1\bar{q} \rightarrow g_2) d\sigma(qg_1g_2\bar{q} \rightarrow g_3 \cdots g_n)] \end{aligned} \quad (56)$$

Eq.(55, 56) is just the dipole radiation formula in the Lund Dipole Cascade Model [9]. In a more general situation the natural ordering variable is transverse momentum, and it is also possible to go beyond the eikonal approximation so that the Altarelli-Parisi splitting functions are properly reproduced for collinear emissions. In ref. [9] it is also demonstrated that the dipole formulation reproduces the angular ordering constraint due to soft gluon interference [10]. This restricts interference effects and implies that gluon emission is a local process, in which only a limited set of gluons contribute to the emission of a softer gluon (where soft should mean in the rest frame of the parent set).

We have derived the Lund dipole cascade model from the H_c approach in the leading order. Now we begin to discuss the next-to-leading order. The general case in this order is rather complicated, and we therefore first discuss the simple cases with two and three gluons. The local character of the dipole cascade emission implies, however, that these results may be relevant also for a more general situation.

For $n = 2$, i.e. when there are two gluons in the final state, we have two singlet chain states $|f_{1,2}\rangle = \{|012\rangle, |021\rangle\}$ (the normalization factor is $N_c^{-3/2}$). We have $NL' = \{(012), (021)\}$, $NL_1 = NL = NL'$ and $NL_2 = \overline{NL} = \{\}$. There are two types of inner products between a chain state and a trace state in $|H_c\rangle$: $\langle 012|Tr(012)\rangle = N_c^3 - 2N_c + \frac{1}{N_c} \approx N_c^3$ and $\langle 012|Tr(021)\rangle = -N_c + \frac{1}{N_c} \approx -N_c$. Note that we have used a simplified notation for the chain state and trace state, e.g. $|01234\rangle \equiv |1_{01}1_{12}1_{23}1_{34}1_{40}\rangle$ and $Tr(01234) \equiv Tr(QG_1G_2G_3G_4)$ etc.. If we use orthogonalized chain states as shown in Eq.(31), we have:

$$\begin{aligned} |(012)'\rangle &= (1 + \frac{C_1}{N_c^2})|012\rangle - \frac{C_2}{N_c^2}|021\rangle \\ |(021)'\rangle &= (1 + \frac{C_1}{N_c^2})|021\rangle - \frac{C_2}{N_c^2}|012\rangle \end{aligned} \quad (57)$$

From Eq.(34) and taking into account the normalization factor for chain states, we obtain:

$$\begin{aligned} &|\langle (012)'|H_c\rangle|^2 + |\langle (021)'|H_c\rangle|^2 \\ &= \frac{1}{4}N_c^3 [(1 + 2\frac{C_1}{N_c^2})(|D^{12}|^2 + |D^{21}|^2) - \frac{4(C_2+1)}{N_c^2} Re(D^{12} \cdot D^{21*})] \\ &= N_c^3 g_s^4 \{ (1 + 2\frac{C_1}{N_c^2})(0, \bar{0})_1 [(1, \bar{0})_2 + (0, 1)_2] \\ &\quad - \frac{4(C_2+1)}{N_c^2} (0, \bar{0})_1 [(0, \bar{0})_2 - (0, 1)_2 - (1, \bar{0})_2] \} \end{aligned} \quad (58)$$

where we have used a simplified notation for the antenna term, for example, $(0, 1)_2 \equiv (p, k_1)_{k_2}$ and $(1, \bar{0})_2 \equiv (k_1, p')_{k_2}$ etc.. We see that the magnitude of $(1, \bar{0})_2$ and $(0, 1)_2$ in leading order obtains a correction term which is of order $1/N_c^2$. There is also a negative term $(0, \bar{0})_2$ from the interference which is absent in leading order. This term corresponds to the emission rate of gluon 2 from the dipole $q\bar{q}$.

For $n = 3$, there are six chain states:

$$\{|f_i\rangle, i = 1 \cdots 6\} = \{(0123), (0132), (0213), (0312), (0231), (0321)\} \quad (59)$$

Since $\langle f_i | f_j \rangle = 1/N_c^2$ for any $i \neq j$, all these states belong to the same NL' . NL_1 and NL_2 are given by:

$$\begin{aligned} NL_{1,2}(0123) &= \{(0213), (0132)\}, \{(0321)\} \\ NL_{1,2}(0132) &= \{(0312), (0123)\}, \{(0231)\} \\ NL_{1,2}(0213) &= \{(0123), (0231)\}, \{(0312)\} \\ NL_{1,2}(0312) &= \{(0132), (0321)\}, \{(0213)\} \\ NL_{1,2}(0231) &= \{(0321), (0213)\}, \{(0132)\} \\ NL_{1,2}(0321) &= \{(0231), (0312)\}, \{(0123)\} \end{aligned} \quad (60)$$

and \overline{NL} is obtained by $NL' - NL_1 - NL_2$. We can write the above equation in another form. According to Eq.(60), the following pairs of orders for chain states belong to NL_1 , NL_2 and \overline{NL} , respectively:

$$NL_1 : \quad \{(0123), (0213)\}, \{(0123), (0132)\}, \{(0132), (0312)\} \\ \{(0213), (0231)\}, \{(0312), (0321)\}, \{(0231), (0321)\} \quad (61)$$

$$NL_2 : \quad \{(0123), (0321)\}, \{(0132), (0231)\}, \{(0213), (0312)\} \quad (62)$$

$$\overline{NL} : \quad \{(0123), (0312)\}, \{(0123), (0231)\}, \{(0312), (0231)\} \\ \{(0321), (0132)\}, \{(0321), (0213)\}, \{(0132), (0213)\} \quad (63)$$

According Eq.(61-63), the leading contribution involves

$$\begin{aligned} \sum_{all P} |D^P|^2 &= 2^3 g_s^6 (0, \overline{0})_1 \{ (1, \overline{0})_2 [(0, 1)_3 + (1, 2)_3 + (2, \overline{0})_3] \\ &\quad + (0, 1)_2 [(0, 2)_3 + (2, 1)_3 + (1, \overline{0})_3] \} \end{aligned} \quad (64)$$

The next-to-leading contribution consists of three parts which are from NL_1 , NL_2 and \overline{NL} respectively. The NL_1 contribution corresponds to the following sums:

$$\begin{aligned} \sum_{P \in NL_1} Re(D^P D^{P'*}) &= 2 g_s^6 (0, \overline{0})_1 \{ -I_1(0, 1, \overline{0}|2) I_1(1, \overline{0}, 2|3) \\ &\quad - I_1(0, 1, \overline{0}|2) I_1(1, 0, 2|3) + 2(1, \overline{0})_2 I_1(1, 2, \overline{0}|3) + 2(1, \overline{0})_2 I_1(0, 1, 2|3) \\ &\quad + 2(0, 1)_2 I_1(2, 1, \overline{0}|3) + 2(0, 1)_2 I_1(0, 2, 1|3) \} \end{aligned} \quad (65)$$

where we the interference pattern I_1 is defined by:

$$\begin{aligned} I_1(g_1, g_2, g_3|g_4) &\equiv (g_1, g_3)_{g_4} - (g_1, g_2)_{g_4} - (g_2, g_3)_{g_4} \\ I_1(g_3, g_2, g_1|g_4) &= I_1(g_1, g_2, g_3|g_4) \end{aligned} \quad (66)$$

where g_i is the label for gluon i . The NL_2 contribution corresponds to

$$\begin{aligned} \sum_{P \in NL_2} Re(D^P D^{P'*}) &= 2 g_s^6 (0, \overline{0})_1 I_1(0, 1, \overline{0}|2) \\ &\quad \cdot [I_1(0, 2, \overline{0}|3) + I_1(0, 1, \overline{0}|3) - 2(1, 2)_3] \end{aligned} \quad (67)$$

The \overline{NL} contribution corresponds to

$$\begin{aligned} & \sum_{P \in \overline{NL}} \text{Re}(D^P D^{P'*}) \\ &= 2 g_s^6 (0, \overline{0})_1 \{ 2 (1, \overline{0})_2 I_2(0, 1, 2, \overline{0}|3) + 2 (0, 1)_2 I_2(0, 2, 1, \overline{0}|3) \\ & - I_1(0, 1, \overline{0}|2) [I_1(1, 2, \overline{0}|3) + I_1(0, 1, 2|3) + I_1(0, 2, 1|3) + I_1(2, 1, \overline{0}|3)] \} \end{aligned} \quad (68)$$

where the interference pattern I_2 is defined by:

$$I_2(g_1, g_2, g_3, g_4|g_5) \equiv (g_1, g_4)_{g_5} + (g_2, g_3)_{g_5} - (g_1, g_3)_{g_5} - (g_2, g_4)_{g_5} \quad (69)$$

Substituting the leading and next-to-leading contribution given in Eq.(64-68) into Eq.(34), we can obtain the cross section for chain states up to next-to-leading order.

In a strongly ordered gluon cascade the emission amplitude has an eikonal form. In this case, we have shown that in leading order in N_c , the cross section of $e^+e^- \rightarrow q\bar{q} + ng$ can be decomposed into $n!$ independent or incoherent parts and each part represents the contribution from the chain state with a specific order of gluons. The cross section of a singlet chain state exactly corresponds to the emission of gluons by a specific sequence of dipoles. The sequence is determined by the gluon order of the chain state. The softest gluon is emitted independently and in an equal emission probability by all possible dipoles each of which is stretched by two adjacent harder gluons, and each dipole corresponds to a specific chain state. Hence to sum over all possible chain states is the same thing as to sum over all possible dipole sequences. In next-to-leading order, the emission probability acquires a $O(1/N_c^2)$ correction and may not be necessarily equal for different neighbor-gluon dipoles. Furthermore there appear dipoles stretched by non-adjacent gluons which are separated by only one extra gluon. The next-to-leading corrections arise from interferences of two D -functions with their gluon orders in the $NL_{1,2}$ or \overline{NL} set. For two D -functions belonging to $NL_{1,2}$, only one interference pattern I_1 is relevant. The other interference pattern I_2 is associated with D -functions belonging to \overline{NL} . I_1 is related to three consecutive gluons and I_2 to four gluons which are not necessarily consecutive. In summary, each next-to-leading correction brought by the interference of a pair of D -functions, say, D^P and $D^{P'}$, can be regarded as a small perturbation to the dipole sequence which corresponds to $|D^P|^2$ or $|D^{P'}|^2$ by having dipoles formed by non-adjacent gluons while keeping the rest of the dipole sequence same as that corresponding to $|D^P|^2$ or $|D^{P'}|^2$. Compared to higher-order corrections, the next-to-leading correction causes the least perturbation to the dipole sequence of the leading order.

VII. SUMMARY AND CONCLUSION

The phenomenological color flow picture, commonly used in the Lund model and the cluster model, is to assign the color connection of a final parton system. In these models, for an $e^+e^- \rightarrow q\bar{q} + ng$ event, the neutral color flow is definitely determined and begins at the quark, connects each gluon one by one in a certain order, and ends at the antiquark. Each flow piece spanned between two partons is color-neutral and its hadronization is treated in a way similar to a $q\bar{q}$ singlet system. The present hadronization models work successfully, which shows that this picture is a good approximation to the real world. In this paper we use the method of color Hamiltonian, a strict formulation developed from PQCD, to study

the structure of chain states in $e^+e^- \rightarrow q\bar{q} + ng$ for finite $N_c = 3$ and in the large N_c limit. For large N_c these states correspond to well-defined color topologies. They just correspond to the phenomenological neutral color flow. Therefore we may expect that the fraction of the non-chain state is an estimate of the fraction of events, where color reconnection is possible. It is also conceivable that color structures, where one gluon is connected to more than two other gluons, may result in more complicated hadron configurations. (The problem of finding experimental observables for these final states is, however, beyond the scope of this paper.)

There are $n!$ chain states, each of which corresponds to a specific order of n gluons. When N_c is 3, as in the real world, any two different chain states are not orthogonal to each other. To derive the total fraction for chain states we must orthogonalize them. We give two types of recipes of orthogonalization: one is to symmetrize the original chain states, the other is to find the transformation matrix which slightly differs from the unit matrix, by exploiting the fact that every two different chain states are approximately orthogonal up to order $\frac{1}{N_c^2}$. As an example, we give the numerical result for the rate of chain states in $e^+e^- \rightarrow q\bar{q}g_1g_2$. The result is shown as a function of the cutoff y_{cut} . The rate decreases slowly, from 0.72 to 0.67, as y_{cut} varies from 10^{-4} to 10^{-2} . These values are smaller than the rate 0.83 obtained by neglecting the kinematic interference contribution. The difference is not large, which implies that kinematic interference terms are less important than non-interference ones, but not negligibly small. Therefore we may expect that the fraction of non-chain states is an estimate of the fraction of events where color reconnection is possible.

Similar to $n!$ singlet chain states, there are also $n!$ terms in the color Hamiltonian, where each term corresponds to an order between n gluons. Up to $O(\frac{1}{N_c^2})$ we prove, with the help of a diagram technique in section III, that there exists a *one-to-one* correspondence between a chain state and the term D^P in H_c , with the same order of gluon labels. This means that when computing the fraction of a chain state $|f\rangle$ with a specific order of gluon connections, up to $O(1/N_c^2)$ we only need to consider the contribution from the term in H_c where the gluon labels are in the same order, and those terms for which the order of gluons is most close to it.

Finally we give the explicit form for the D function and H_c in a special case, the case of soft gluon bremsstrahlung. In soft gluon bremsstrahlung with gluons strongly ordered in energy or transverse momentum, the emission amplitude has an eikonal form. In this case, we have shown that in leading order in N_c , the cross section of $e^+e^- \rightarrow q\bar{q} + ng$ can be decomposed into $n!$ independent or incoherent parts and each part represents the contribution from the chain state with a specific order of gluons. We also give the next-to-leading order corrections for $e^+e^- \rightarrow q\bar{q}g_1g_2$ and $e^+e^- \rightarrow q\bar{q}g_1g_2g_3$. The corrections arise from interferences of two D -functions with their gluon orders in the $NL_{1,2}$ or \overline{NL} set. Each next-to-leading correction brought by the interference of a pair of D -functions, say, D^P and $D^{P'}$, can be regarded as a small perturbation to the dipole sequence which corresponds to $|D^P|^2$ or $|D^{P'}|^2$ by having dipoles formed by non-adjacent gluons, while keeping the rest of the dipole sequence same as that corresponding to $|D^P|^2$ or $|D^{P'}|^2$. The next-to-leading correction causes the least perturbation to the dipole sequence of the leading order compared to higher-order corrections.

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FIGURES

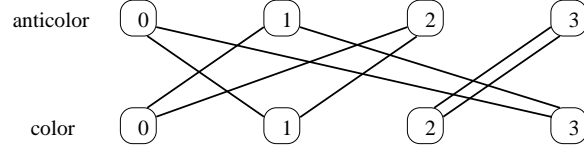


FIG. 1. The diagram for calculating $\langle 1_{q1}1_{12}1_{23}1_{3\bar{q}} \mid 1_{q2}1_{23}1_{31}1_{1\bar{q}} \rangle$. The number of closed paths is $l = 2$. Thus the result is N_c^2 .

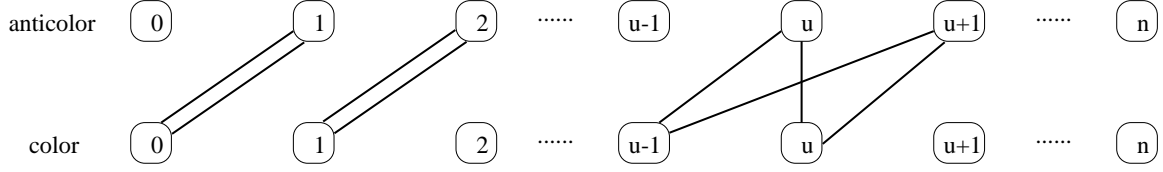


FIG. 2. The diagram for $\langle 1_{q1}1_{12}1_{23} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}} \mid 1_u1_{q1}1_{23} \cdots 1_{u-2,u-1}1_{u-1,u+1} \cdots 1_{n\bar{q}} \rangle$. There are $(n + 1 - 2)$ double-line loops plus one loop connecting color $u - 1, u$ and anticolor $u, u + 1$.

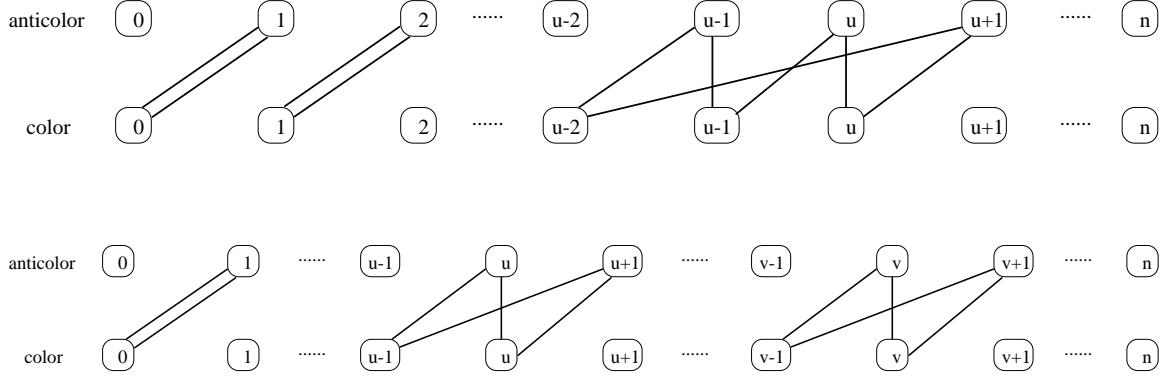


FIG. 3. The diagram for $\langle 1_{q1}1_{12} \cdots 1_{i,i+1} \cdots 1_{n\bar{q}} \mid 1_{u1}1_{u2}(1_{qv1}1_{v1v2} \cdots 1_{v_iv_{i+1}} \cdots 1_{v_{n-k}\bar{q}}) \rangle$. (a) u_1 and u_2 are neighbors. (b) u_1 and u_2 are not neighbors.

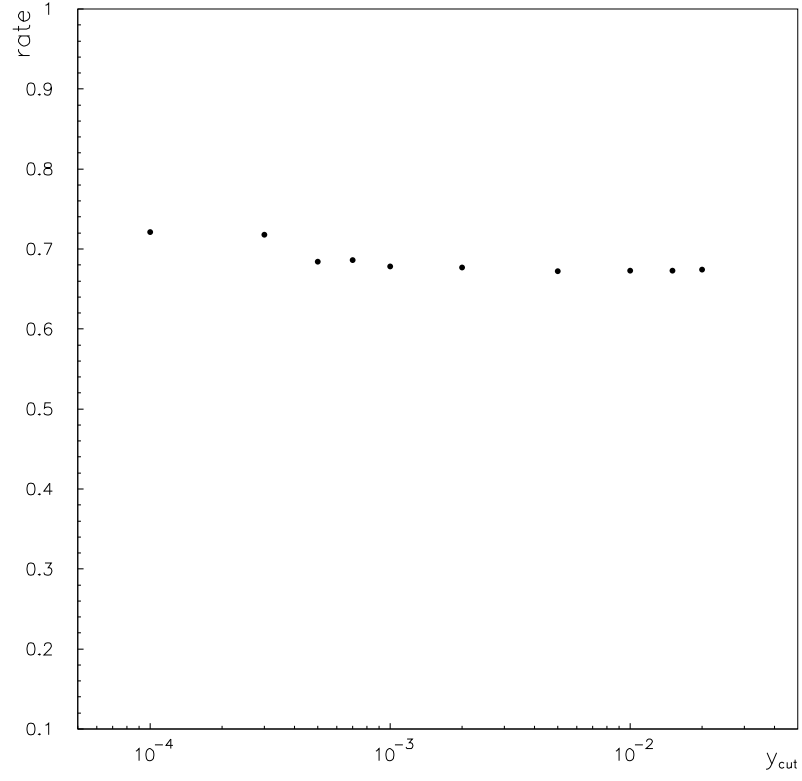


FIG. 4. The numerical result for the fraction of chain states in e^+e^- annihilation into two gluons.

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